Goal: understand $\lim_{g\to\infty} H_*(B\mathrm{Diff}^+\Sigma_{g,1})$,
prove the Mumford conjecture:
$$\lim_{g\to\infty} H^*(B\mathrm{Diff}^+\Sigma_{g,1};\mathbb{Q}) = \mathbb{Q}[e_1^{\in H^2}, e_2^{\in H^4}, e_3^{\in H^6}, \ldots]$$

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Stable homology through scanning
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Outline:

$B\mathrm{Diff}^+\Sigma_g$ = subsets of $\mathbb{R}^\infty$ diffeomorphic to Σ_g

$\coprod_{g\geq 0} B\mathrm{Diff}^+\Sigma_g$ = subsets of $\mathbb{R}^\infty$ diffeomorphic to a closed surface

approximated by $S(0,N)$ = subsets of $\mathbb{R}^N$ diffeomorphic to a closed surface

$S(N,N)$ = properly embedded 2-dimensional manifolds in $\mathbb{R}^N$, topologized so manifolds can disappear at infinity (not necessarily compact, connected, finite type, etc.)

$\vdots$

$S(k,N)$ = subspace of $S(N,N)$ consisting of properly embedded 2-dimensional manifolds contained in $\mathbb{R}^k\times(0,1)^{N-k}$

$\vdots$

$S(1,N)$ = properly embedded 2-dimensional manifolds contained in $\mathbb{R}\times(0,1)^{N-1}$

$S(0,N)$ = 2-dimensional closed manifolds in $(0,1)^N$

Problem: operation in $S(0,N)$ is disjoint union: ,
and we want it to be connected sum. 

We will redefine, get $\Sigma(0,N)$ s.t. operation is connected sum and $\lim_{N\to\infty}\Sigma(0,N) = \coprod_{g\geq 0} B\mathrm{Diff}^+\Sigma_{g,1}$

Relaxation principle:

$\boxed{B\Sigma(0,N) = S(1,N)}$ ← Lecture 7

$\boxed{\begin{array}{l} BS(1,N) = S(2,N) \\ \vdots \\ BS(N-1,N) = S(N,N)\end{array}}$ ← today

need $\pi_0(S(k,N))$ to be a group

$H_*(\Sigma(0,N))[\pi_0^{-1}] \simeq H_*(\Omega S(1,N))$

$S(1,N) \simeq \Omega S(2,N)$

$\vdots$

$S(N-1,N) \simeq \Omega S(N,N)$

Zooming: $\boxed{S(N,N) \simeq \mathrm{Aff}_2^+(\mathbb{R}^N)}$, (the one-point compactification of) the space of oriented affine 2-planes in $\mathbb{R}^N$

Combining these, we have $H_*(\Sigma(0,N))[\pi_0^{-1}] \simeq H_*(\Omega^N\mathrm{Aff}_2^+(\mathbb{R}^N))$, and letting $N\to\infty$,

$$\lim_{g\to\infty} H_*(B\mathrm{Diff}^+\Sigma_{g,1}) = H_*(\Sigma(0,\infty))[\pi_0^{-1}] = H_*(\Omega^\infty\mathrm{Aff}_2^+(\mathbb{R}^\infty))$$

$$\boxed{H^*(\Omega^\infty\mathrm{Aff}_2^+(\mathbb{R}^\infty);\mathbb{Q}) = \mathbb{Q}[e_1,e_2,e_3,\ldots]}$$

Lecture 8: why is this the cohomology?
how do these classes translate to characteristic classes of surface bundles?

$\pi_0(S(k,N))$ is a group for $k>0$

Let $d=2$, so that $S(k,N)$ is the space of d-manifolds in $\mathbb{R}^N$, allowed to go to infinity in k directions.

We will give an argument for $d=2$ which works verbatim for all d.

Assume $N>d$. Then:

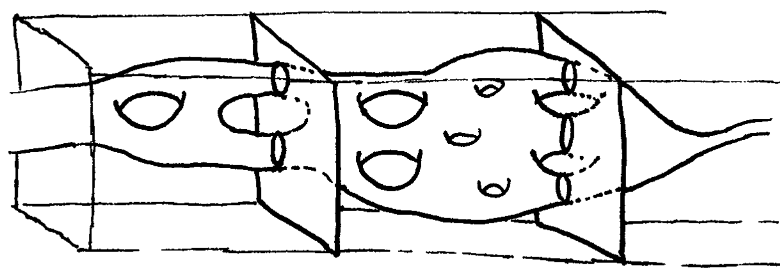
for $0<k\le d$, $\pi_0(S(k,N))=$ the group $\Omega^{SO}_{d-k,N-k}$ of cobordism classes of oriented smooth $(d-k)$-manifolds in $\mathbb{R}^{N-k}$ (with cobordisms embedded in $\mathbb{R}^{N-k}\times I$)

for $d<k$, $\pi_0(S(k,N))=0$.

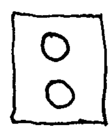
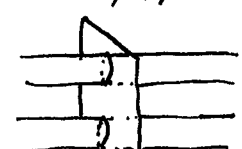
(so for $d=2$, $\pi_0(S(1,N))=0$, $\pi_0(S(2,N))=\mathbb{Z}$, $\pi_0(S(k,N))=0$ for $k>2$)

The map $S(k,N)\to\Omega^{SO}_{d-k,N-k}$ is given by intersecting with a generic $(0,1)^{N-k}$:

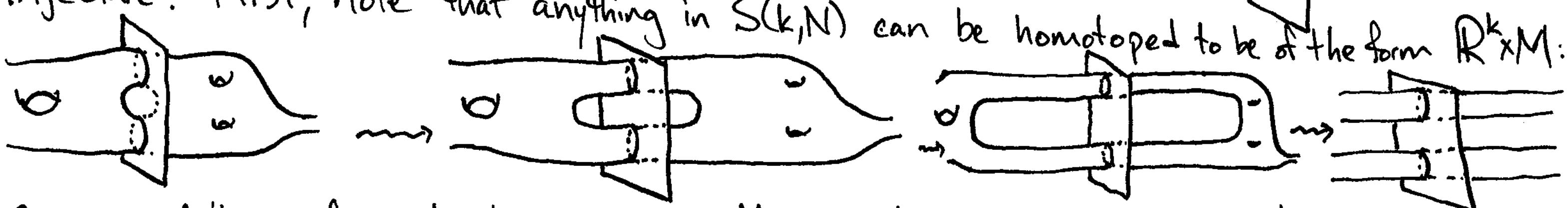
Well-defined? If we intersect with a different $(0,1)^{N-k}$, the slab between them is a cobordism embedded in $(0,1)^{N-k}\times I$.



Surjective? for any $[M]\in\Omega^{SO}_{d-k,N-k}$, we can take $\mathbb{R}^k\times M\in S(k,N)$:

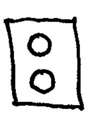
$M=$  $\mathbb{R}^k\times M=$ 

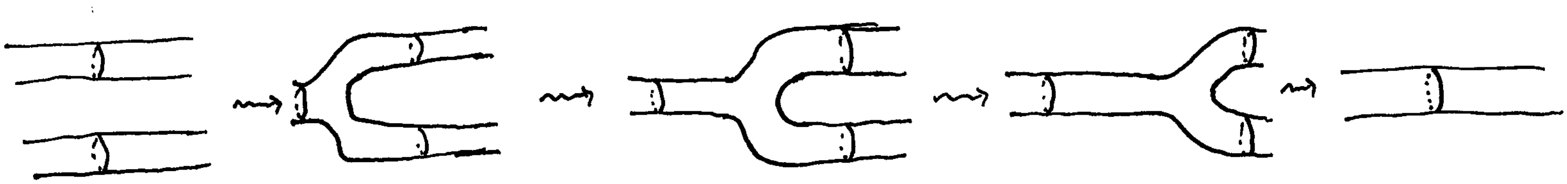
Injective? First, note that anything in $S(k,N)$ can be homotoped to be of the form $\mathbb{R}^k\times M$:



So any failure of injectivity is because M and N are cobordant, but $\mathbb{R}^k\times M$ and $\mathbb{R}^k\times N$ lie in different components of $S(k,N)$.

<u>But</u>, if M and N are cobordant, we can homotope $\mathbb{R}^k\times M$ to $\mathbb{R}^k\times N$ in $S(k,N)$ by <u>zipping</u>.

$M=$ , $N=$ , cobordism$=$ 



$BS(k,N) = S(k+1,N)$ for $k>0$

Caveats:

- we make $S(k,N)$ into a monoid by taking surfaces in $\mathbb{R}^k \times (0,\ell) \times (0,1)^{N-k-1}$ for $\ell>0$, with operation given by juxtaposition in the $k+1^{\text{st}}$ coordinate.
- since $BS(k,N)$, we'd better take only one component $S(k+1,N)_0$; this doesn't affect the outline, since $BS(k,N) = S(k+1,N)_0$ gives $S(k,N) \simeq \Omega S(k+1,N)_0 = \Omega S(k+1,N)$.

$$S(k+1,N)_0 \xleftarrow{\ \simeq\ } S_{\text{walls}}(k+1,N) \xrightarrow{\ \simeq\ } BS(k,N)$$

a point in $S_{\text{walls}}(k+1,N)$ is:

- a surface $S \in S(k+1,N)$,
- $n+1 \geq 1$ walls W_i disjoint from S
- weights $t_i \geq 0$ with $t_0 + \cdots + t_n = 1$

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the map $S_{\text{walls}}(k+1,N) \xrightarrow{\ \simeq\ } BS(k,N)$ is defined just as before: after forgetting what's outside the walls, the walls cut into slabs giving elements

$\in S(k,N)$,

and together with the weights t_i we have coordinates in $S(k,N) \times S(k,N) \times \Delta^2 \subset BS(k,N)$.

Since fibers are contractible, this gives a homotopy equivalence $S_{\text{walls}}(k+1,N) \xrightarrow{\ \simeq\ } BS(k,N)$.

The hard part now is showing that

$S(k+1,N)_0 \xleftarrow{\text{forget walls}} S_{walls}(k+1,N)$ is a homotopy equivalence.

Previously this map was a surjective fibration with contractible fibers (convex combinations of all the walls that can be legally inserted).

Here the fibers are still contractible, but the map is no longer surjective:  is not disjoint from any wall.

The image of $S_{walls}(k+1,N)$ is $S_{disjoint} = S_{disjoint}(k+1,N)$, consisting of surfaces in $S(k+1,N)$ disjoint from at least 1 wall.

We have:

$S(k+1,N)_0 \hookleftarrow S_{disjoint}(k+1,N) \xleftarrow{\simeq} S_{walls}(k+1,N)$ and need to show that $S_{disjoint}(k+1,N) \xhookrightarrow{\simeq} S(k+1,N)_0$

For any surface $S \in S(k+1,N)_0$, we will find a wall W and a homotopy h^W so that $h^W_0 = \mathrm{Id}$ and $h^W_1(S)$ is disjoint from W, giving a path from S to $S_{disjoint}$.

But our construction will additionally satisfy the properties that:

- the same homotopy h^W works for any surface S' near to S
- h^W is the identity outside a narrow slab around W

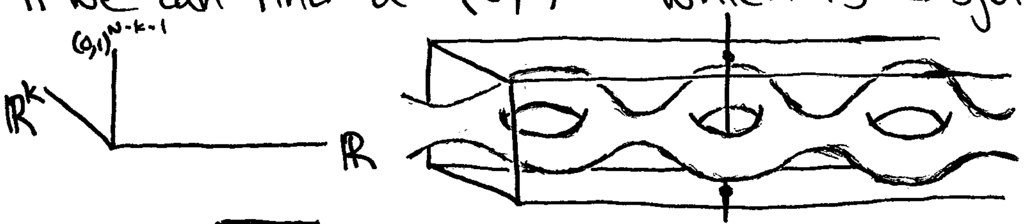
So for any $D^n \to S(k+1,N)_0$ with $\partial D^n \subset S_{disjoint}$, by compactness of D^n we can choose finitely many walls W_i s.t. for each $S \in D^n$, at least one h^{W_i} works.

Since the h^{W_i} are disjointly supported, we can piece them together using a partition of unity on D^n, and get a homotopy h of $D^n \to S(k+1,N)_0$ s.t. $h_1(D^n) \subset S_{disjoint}$.

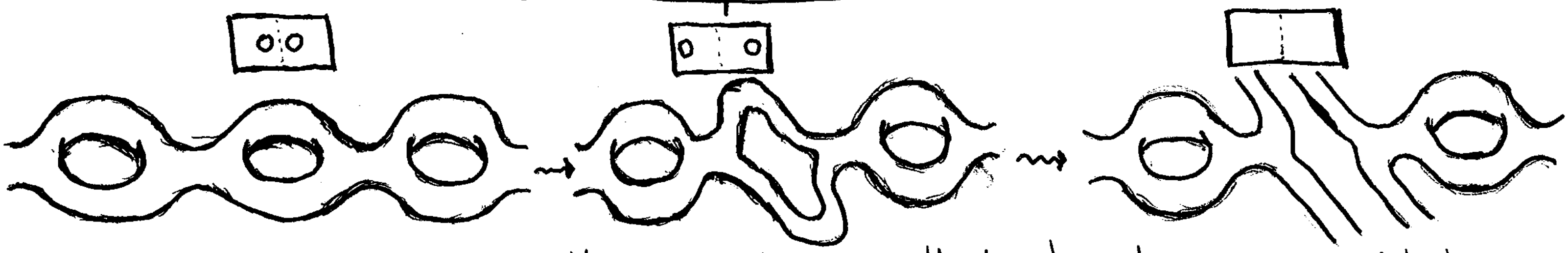
This shows the relative homotopy groups of $(S(k+1,N)_0, S_{disjoint})$ are trivial, so $S_{disjoint} \xhookrightarrow{\simeq} S(k+1,N)_0$ as desired.

How can we find such a wall W and homotopy h^W?

If we can find a $(0,1)^{N-k-1}$ which is disjoint from S,

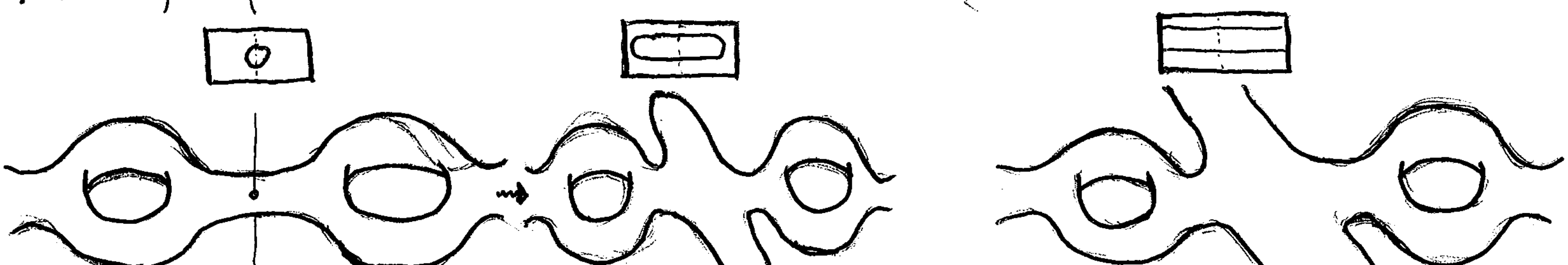


we can push away from it in the $\mathbb{R}^k$ direction:

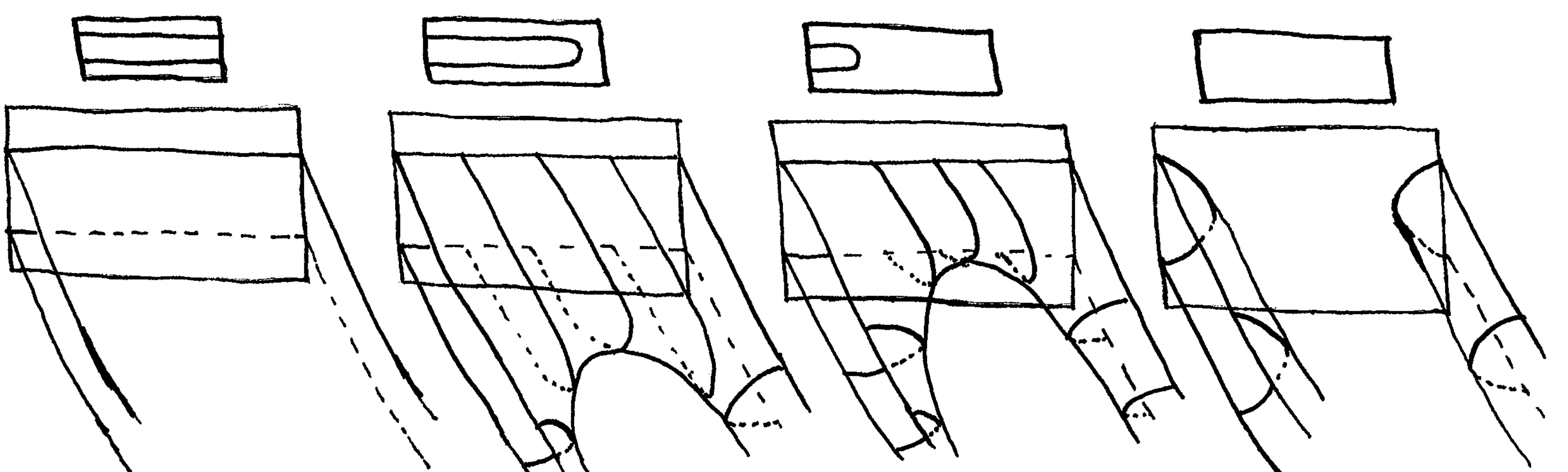


Whenever $k+1>d$, Sard's theorem tells us that almost every $(0,1)^{N-k-1}$ is disjoint from S. So we only need to consider $k+1 \leq d$, which for $d=2$ means only $\underline{S(2,N)}$:

1) First, expand in $\mathbb{R}^k$ direction as before:



2) Now unzip the middle of the product region near this slice (while keeping the rest of the suface fixed):



To do this, we need the intersection $(0,1)^{N-k-1} \cap S =$ 

to bound a manifold, e.g. 

But recall that $S \mapsto (0,1)^{N-k-1} \cap S$ induces the isomorphism $\pi_0(S(k+1,N)) \approx \Omega^{SO}_{d-k-1,N-k-1}$, so what we need is exactly that $S \in S(k+1,N)_0$.